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Instability of a Nielsen-Olesen Vortex Embedded in the Electroweak Theory:‡ I. The Single-Component Higgs Gauge

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Abstract. The stability of an abelian (Nielsen-Olesen) vortex embedded in the electroweak theory against W production is investigated in a gauge defined by the condition of a single-component Higgs field. The model is characterized by the parameters $\beta = (\frac{M_H}{M_Z})^2$ and $\gamma = \cos^2\theta_{\rm w}$ where $\theta_{\rm w}$ is the weak mixing angle. It is shown that the equations for W's in the background of the Nielsen-Olesen vortex have no solutions in the linear approximation. A necessary condition for the nonlinear equations to have a solution in the region of parameter space where the abelian vortex is classically unstable is that the W's be produced in a state of angular momentum m such that $0 > m > -2n$. The integer n is defined by the phase of the Higgs field, $\exp(in\varphi)$. It is shown that, in the region of parameter space (β, γ) where the nonlinear equations have a solution with energy lower than that of the abelian vortex, this vortex is a saddle point of the energy in the space of classical field configurations. Solutions for a set of values of the parameters β and γ in this region were obtained numerically for the case $-m = n = 1$. The possibility of existence of a stationary state for $n = 1$ with W's in the state $m = -1$ was investigated. The boundary conditions for the Euler-Lagrange equations required to make the energy finite cannot be satisfied at $r = 0$. For these values of n and m the possibility of a finite-energy stationary state defined in terms of distributions is discussed.

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Introduction

It has been shown that the Nielsen-Olesen abelian vortex [1] can be embedded [2, 3] in the electroweak SU(2) × U(1) gauge theory [4, 5] in the form of an azimuthal Z field $Z_\varphi(\rho)$ and a lower component of the Higgs field $\Phi_2 = \Phi(\rho)\exp(in\varphi)$, where (ρ, φ) are polar coordinates of the position vector $\boldsymbol{\rho}$ perpendicular to the vortex. The embedded vortex, hereafter denoted the Z_{NO} vortex, is a tube of confined flux of the Z field strength, which reaches a high value at the center of the vortex.

It is known from previous works that, in a strong uniform magnetic field, the electroweak vacuum develops an instability through the interaction of the magnetic field with the anomalous magnetic moment of the W boson, leading to the formation of a W condensate [6, 7, 8]. The magnetic moment interacts similarly with a Z field; hence an instability with ensuing W production can occur if the Z field strength is sufficiently high within a region large enough compared to the Compton wavelength of the W boson. A measure of these conditions is provided, respectively, by the two parameters $\beta = (M_H/M_Z)^2$ and $\gamma = (M_W/M_Z)^2 \equiv \cos^2\theta$, where M_H, M_Z, M_W are the masses of the Higgs, Z and W bosons and θ is the Weinberg mixing angle. One finds qualitatively that the possibility of instability increases with higher β and higher γ.

A quantitative investigation of the stability of the $\mathrm{Z_{NO}}$ vortex for $n = 1$ was performed by James, Perivolaropoulos and Vachaspati [9]. They found numerically that the solution becomes unstable beyond a certain line in the parameter space (β, γ). Their analysis was supplemented with an elegant analytical estimate by Perkins [10], according to which the $\mathrm{Z_{NO}}$ solution is unconditionally unstable for $\gamma > .19$. In particular, the points (β, γ) corresponding to the physical value of the Weinberg angle are inside the region of instability.

In this report we have investigated the problem in a gauge which maintains the simple structure of the $\mathrm{Z_{NO}}$ vortex, defined by the condition that the upper component of the Higgs field vanishes, rather than the gauge used by James et al. [*op. cit.*] and Achúcarro et al. [11] which allows for a two-component Higgs field in the presence of W bosons. These two gauges are actually inequivalent since (for $n = 1$) the gauge invariant quantity $(\Phi_1^\dagger\Phi_1 + \Phi_2^\dagger\Phi_2)$, at $\rho = 0$, is zero in our gauge but non-zero in theirs.

In the first section an ansatz for the fields is presented that preserves the cylindrical symmetry of the energy density. The Euler-Lagrange equations are obtained and the boundary conditions are established.

In Section 2 we consider the equations for W bosons in the background of the configuration of Higgs and Z fields as given by the $\mathrm{Z_{NO}}$ vortex. It is shown that, in this gauge, these equations have no solutions in the linearized form. On the other hand for particular angular momenta of the W's, specified by the condition $-2n < m < 0$ on the phase $\exp(im\varphi)$ of the polar components of the W field, the set of nonlinear equations may admit a solution in a certain domain of the space of parameters (β, γ). It is shown that, in the region of parameter space where the

nonlinear equations have a solution and the energy, calculated to lowest order in the W field, is smaller than that for the $\mathrm{Z_{NO}}$ vortex, this vortex is not a local minimum but a saddle point in the space of classical field configurations.

In Section 3, these equations were solved numerically for the case $n = 1$, $m = -1$ and a set of values of the parameters $\beta = .5, 1$ and $\gamma = .25, .5, 1$. The energy was computed and found in each case to be lower than that for the $\mathrm{Z_{NO}}$ vortex. As remarked already, the field configurations considered here are unrelated by any gauge transformation to those considered in Refs. [9, 11]. Hence the instability regions for the two cases may be different.

The existence of a stationary state with W's, as suggested in Refs. [10, 12, 13], is considered in Section 4. For $m = -1$, one would expect that there exists an analytic solution of the Euler-Lagrange equations for all fields. We found, however, that the boundary conditions cannot be satisfied at $r = 0$. A discussion is given of the possibility of a vortex state with W's defined in terms of distributions.

1 Nonabelian Vortex

We shall investigate the problem of stability of the $\mathrm{Z_{NO}}$ vortex in a gauge fixed by the condition that the Higgs field Φ has a zero upper component and a lower component $\Phi(\rho)\exp(in\varphi)$. For the vortex of the two-dimensional abelian theory, n is a topological winding number defined in terms the total flux of the U(1) gauge-field strength [1]. In a non-abelian model n can no longer be defined in a gauge invariant way. In the above chosen gauge it is given by

$$n = \frac{1}{2\pi i \Phi_0^2} \int (d\Phi^\dagger \wedge d\Phi)$$

in the notation of differential forms, where Φ_0 is the magnitude of the Higgs field at infinity. This expression is invariant only under the electromagnetic U(1) gauge group.

The $\mathrm{Z_{NO}}$ vortex contains an azimuthal Z field Z_φ. One can easily show that, if a radial component depending on the ρ coordinate alone is added to the Z field, the action increases. Therefore, the vortex solution can only be modified by the inclusion of a W field and an electromagnetic gauge potential. The latter can be chosen purely azimuthal by virtue of the residual electromagnetic gauge invariance.

Let g, g' be the coupling constants for the groups SU(2) and U(1) respectively. They are related to the Weinberg angle θ and the electromagnetic charge e by $g \sin\theta = g' \cos\theta = e$. The physical gauge fields are related to the gauge potentials $\mathbf{V}^a$ and $\mathbf{V}'$ associated with the groups SU(2) and U(1) by

$$\begin{aligned} \mathbf{A} &= \mathbf{V}' \cos\theta + \mathbf{V}^3 \sin\theta \ , \\ \mathbf{Z} &= -\mathbf{V}' \sin\theta + \mathbf{V}^3 \cos\theta \ , \\ \mathbf{W} &= \tfrac{1}{\sqrt{2}}(\mathbf{V}^1 - i\mathbf{V}^2) \ . \end{aligned}$$

Let us introduce a dimensionless vector

$$\mathbf{r} = \boldsymbol{\rho}\,\Phi_0 g/(\sqrt{2}\cos\theta) \equiv \boldsymbol{\rho} M_Z \ ,$$

with polar coordinates r, φ, and a set of functions s, X, Y, Z defined by

$$\Phi = \Phi_0 s(r) e^{in\varphi} \ ,$$

$$\begin{array}{ll} \frac{1}{\sqrt{2}} V_\varphi^3 \cos\theta = \Phi_0 Y(r) \ , & \frac{1}{\sqrt{2}} V'_\varphi \sin\theta = \Phi_0 X(r) \ , \\ \frac{1}{\sqrt{2}} Z_\varphi = \Phi_0 (Y - X) = \Phi_0 Z(r) \ , & \frac{1}{\sqrt{2}} A_\varphi = \Phi_0 (Y \tan\theta + X \cot\theta) \ . \end{array}$$

In order to preserve the vortex cylindrical symmetry the $\mathbf{W}$ field must be of the form

$$\mathbf{W}\cos\theta = \Phi_0 [u(r)\mathbf{e}_r + iv(r)\mathbf{e}_\varphi] \exp(im\varphi) \ .$$

It can be shown that the functions u and v may be chosen real without loss of generality. It is also convenient to use a set of auxiliary fields

$$\begin{array}{rcl} y & = & Y - \frac{m}{2r} \\ x & = & X - \frac{m}{2r} - \frac{n}{r} \\ z & = & Z + \frac{n}{r} = y - x \end{array}$$

and the parameters

$$\beta = \left(\frac{M_H}{M_Z}\right)^2, \quad \gamma = \left(\frac{M_W}{M_Z}\right)^2 = \cos^2\theta \ .$$

The energy density in terms of these fields and the new variables $\mathbf{r}$ takes the form

$$\begin{array}{rcl} \mathcal{H} & = & \Phi_0^2 \left\{ (s')^2 + ((y-x)s)^2 + \frac{1}{4}\beta(s^2-1)^2 + \frac{1}{\gamma}(v' + \frac{v}{r} + 2yu)^2 \right. \\ & + & \left. \frac{1}{\gamma}\Big(y' + \frac{y}{r} - 2uv\Big)^2 + \frac{1}{1-\gamma}\Big(x' + \frac{x}{r}\Big)^2 + (u^2+v^2)s^2 \right\} \ . \end{array} \tag{1.1}$$

The vortex energy per unit length is then given by $\int \mathcal{H}\, d^2\mathbf{r}$. The expression for $\mathcal{H}$ is invariant under the combined substitutions $y \to -y,\ x \to -x,\ v \to -v$ (charge conjugation invariance) so it is sufficient to consider positive values of n.

The Euler-Lagrange equations for the fields are

$$s'' + \frac{s'}{r} - [u^2 + v^2 + (y-x)^2 + \frac{\beta}{2}(s^2-1)]s = 0 \tag{1.2}$$

$$x'' + \frac{x'}{r} - \frac{x}{r^2} + (1-\gamma)(y-x)s^2 = 0 \tag{1.3}$$

$$y'' + \frac{y'}{r} - \frac{y}{r^2} - 2(2v'u + vu' + \frac{1}{r}vu) - 4yu^2 - \gamma(y-x)s^2 = 0 \tag{1.4}$$

$$v'' + \frac{v'}{r} - \frac{v}{r^2} + 2(2y'u + yu' + \frac{1}{r}yu) - 4vu^2 - \gamma v s^2 = 0 \tag{1.5}$$

$$v'y - vy' + (2y^2 + \frac{\gamma}{2}s^2 + 2v^2)u = 0 \ . \tag{1.6}$$

We shall also study the equations for W's produced in the background of the Z_{NO} vortex. For this purpose we have to consider the Euler-Lagrange equations for the fields u, v, and A_φ, with s, z fixed at their Z_{NO} values. It is convenient to use y, u, v as new independent variables, so that $x = y - z$. The equations for u and v will be the same, (1.5) and (1.6), but the equation for y must be modified:

$$y'' + \frac{y'}{r} - \frac{y}{r^2} - 2(1-\alpha)(2v'u + vu' + \frac{vu}{r} + 2yu^2) - \gamma(y-x)s^2 = 0, \tag{1.7}$$

where $\alpha = \gamma$ if A_φ is allowed to vary or $\alpha = 1$ if A_φ is kept at its Z_{NO} value $A_\varphi = 0$. This equation coincides with the previous equation (1.4) if one sets $\alpha = 0$. Therefore the new equation can be used in the three cases with the appropriate values of α. Note that in all cases α is a non-negative parameter.

From the last three equations one obtains the integrability condition

$$\frac{d}{dr}[ru(\gamma s^2 + 4\alpha v^2)] + 2r(\gamma x s^2 + 4\alpha y u^2)v = 0 \ . \tag{1.8}$$

The full system of equations can be reduced to 4 independent second-order differential equations by solving (1.6) for u and (1.8) for u'.

In order to obtain solutions with a finite energy per unit length of the vortex, the following boundary conditions are imposed.

Boundary conditions near $r = 0$:

$$s = s_0 r^n, \quad x = -\tfrac{2n+m}{2r} + x_0 r, \quad y = -\tfrac{m}{2r} + y_0 r, \quad v = v_0 r^k, \quad u = u_0 r^k \tag{1.9}$$

with $k > -\frac{1}{2}$. Inserting these into the equations, one finds that solutions exist near the origin in the following three cases.

$$\begin{array}{llll} \text{a)} & m = 0, & k = 1, & (1+n)u_0 = n v_0 \ . \\ \text{b)} & m = k+1, & k = 0, 1, 2 \ldots, & u_0 = v_0 \ . \\ \text{c)} & m = -k - 2n - 1, & k = 0, 1, 2, \ldots, & m u_0 = -(m+2n) v_0 \ . \end{array} \tag{1.10}$$

These are the boundary conditions valid for the case $\alpha = 0$ corresponding to the variational problem for all the fields or for the same equations linearized with respect to u, v. For the case of the new equations with $\alpha > 0$, the boundary conditions will still be the same for values of n, m such that $k > n$. On the other hand, assuming that $k \leq n$, one finds that the equations near $r = 0$ can only be satisfied if

$$\begin{array}{lll} \text{d)} & m = 0, \quad k = 1, \quad n = 1, & (\gamma s_0^2 + 4\alpha v_0^2) u_0 = \frac{\gamma}{2} s_0^2 v_0 \ . \\ \text{e)} & m \neq 0, \quad k = 0, & v_0 = m u_0 \ . \end{array} \tag{1.11}$$

We remark however that, for $k = 0$, continuity of $\mathbf{W}$ at the origin restricts the value of m to be $m = \pm 1$.

Boundary conditions near $r = \infty$:

Depending on the values of β and γ, different terms in the asymptotic equations are responsible for the leading exponential behavior at large r. Write $s = 1 - f$. Assuming that $0 < \gamma \leq 1$, we consider first the simplest case:

i) $\beta \leq 4\gamma$, $4\gamma > 1$

The asymptotic field equations are, to leading order,

$$\begin{aligned}
1.2' &: \quad f'' + \tfrac{f'}{r} - \beta f = 0 \\
1.3' &: \quad z'' + \tfrac{z'}{r} - \tfrac{z}{r^2} - z = 0 \\
1.6' &: \quad (v' + \tfrac{v}{r})\tfrac{y_1}{r} + (2(\tfrac{y_1}{r})^2 + \tfrac{1}{2}\gamma)u = 0 \\
1.7' &: \quad y'' + \tfrac{y'}{r} - \tfrac{y}{r^2} - \gamma z = 0 \\
1.8' &: \quad u' + \tfrac{u}{r} + 2y_1\tfrac{v}{r} = 0 \ ,
\end{aligned}$$

where equation (1.3') was obtained by subtracting (1.3) from (1.7). Differentiating (1.8'), and using (1.6') and (1.8') to eliminate v' and v, one obtains an asymptotic equation for u,

$$u'' + 3\frac{u'}{r} + \frac{u}{r^2} - (4(\frac{y_1}{r})^2 + \gamma)u = 0 \ . \tag{1.12}$$

From equations (1.2', 1.3', 1.7', 1.12) and (1.8'), one finds the following approximate solutions for large r:

$$\begin{aligned}
f(r) &\sim f_1 K_0(\sqrt{\beta}\, r) \\
z(r) &\sim z_1 K_1(r) \\
y(r) &\sim \tfrac{y_1}{r} + \gamma z \\
u(r) &\sim -v_1 \tfrac{2y_1}{r} K_{|2y_1|}(\sqrt{\gamma}\, r) \\
v(r) &\sim v_1 \tfrac{d}{dr} K_{|2y_1|}(\sqrt{\gamma}\, r) \ .
\end{aligned} \tag{1.13}$$

The parameter y_1 is related to the total flux of the electromagnetic field,

$$\oint_\infty \mathbf{A} \cdot \boldsymbol{d\rho} = \frac{2\pi}{e}(2y_1 + 2\gamma n + m) \ .$$

For values of β, γ which do not satisfy condition (i) above, the expressions for z (or y) and f must be modified as follows.

ii) If $4\gamma \leq 1$,

then the asymptotic expression for z in the set of equations (1.2–1.6), or for y in the set of equations (1.5–1.7), has to be modified. Making use of the Green function for

the Laplace operator in two dimensions let us define:

$$\begin{aligned}\zeta(r) &= -\frac{1}{2\pi}\int K_0(|\mathbf{r}-\mathbf{r}'|)\cos(\varphi-\varphi')\, j(r')\, d^2\mathbf{r}' \ , \\ j(r) &\equiv 2\left(2v'(r)u(r)+v(r)u'(r)+\tfrac{1}{r}v(r)u(r)+\tfrac{2y_1}{r}[u(r)]^2\right) . \end{aligned} \tag{1.14}$$

Then, for the set of equations (1.2–1.6), one would have

$$z = z_1 K_1(r) + \zeta(r) \ , \tag{1.15}$$

while for the set (1.5–1.7) one would have

$$y = \frac{y_1}{r} + \gamma z_1 K_1(r) + (1-\alpha)\zeta(r) \ . \tag{1.16}$$

iii) If $\beta \geq 4\gamma$ or $\beta \geq 4$,

then the expression for f has to be modified. Similarly to case (ii) one finds

$$f(r) = f_1 K_0(\sqrt{\beta}\, r) + \frac{1}{2\pi}\int K_0(\sqrt{\beta}|\mathbf{r}-\mathbf{r}'|)\left(z(r')^2+u(r')^2+v(r')^2\right) d^2\mathbf{r}' \ . \tag{1.17}$$

In the integrands of (1.14, 1.17), u and v are given by their asymptotic expressions (1.13), and in equation (1.17), z is given by (1.13) or (1.15) as prescribed by the value of γ.

The result (1.17) means that the leading asymptotic behavior of the Higgs field is determined, in case (iii), not by the Higgs mass but by twice the W boson mass or, in the limit of zero W fields, by twice the Z mass. This finding agrees with the expressions obtained in a recent reanalysis of the Nielsen-Olesen vortex [14].

The asymptotic expressions involve four parameters, f_1, z_1, y_1 and v_1. Together with the four boundary parameters s_0, x_0, y_0 and v_0 at $r=0$, the number of unknowns equals the rank of the system of differential equations. Therefore, if a solution to the equations exists, then all parameters would be determined by imposing the respective boundary conditions at $r=0$ and at a value $r=r_1 \gg 1$.

We shall now investigate the question of existence of a solution with these boundary conditions. Let us introduce the functions $V=v/y$ and $U=ru(\gamma s^2+4\alpha v^2)$. The equations for U and V are

$$V' + 2(1+\tfrac{1}{y^2}(v^2+\tfrac{\gamma}{4}s^2))u = 0 \ , \tag{1.18}$$

$$U' + 2r(\gamma x s^2 + 4\alpha y u^2)v = 0 \ . \tag{1.19}$$

Multiplying (1.18) by U, (1.19) by V, and adding the equations one obtains

$$(UV)' + 2r\left\{\left([1+\tfrac{1}{y^2}(v^2+\tfrac{\gamma}{4}s^2)]\,(\gamma s^2+4\alpha v^2)+4\alpha v^2\right)u^2 + \tfrac{x}{y}\gamma s^2 v^2\right\} = 0 \ . \tag{1.20}$$

The behavior of $UV = ruv(\gamma s^2 + 4\alpha v^2)/y$ as $r \to 0$ and $r \to \infty$ is as follows:

$$
\begin{aligned}
r \to 0: &\qquad UV \sim \begin{cases} r^2, & m \neq 0, \quad k = 0, \quad \alpha \neq 0 \\ r^{(2n+2)}, & m \neq 0, \quad k = 0, \quad \alpha = 0 \\ r^{(2n+2k+2)}, & m \neq 0, \quad k > 0 \\ r^{(2n+2)}, & m = 0, \quad k = 1 \ . \end{cases} \\
r \to \infty: &\qquad UV \sim \exp(-2\sqrt{\gamma}\, r) \ .
\end{aligned}
\tag{1.21}
$$

If the solution to the equations is such that y does not have a finite zero, then integrating (1.20) from 0 to ∞ one obtains

$$
\int 2r\, dr \Big([1 + \frac{1}{y^2}(v^2 + \frac{\gamma}{4}s^2)]\, (\gamma s^2 + 4\alpha v^2) + 4\alpha v^2 \Big) u^2 + \int 2r\, dr\, \frac{x}{y}\gamma s^2 v^2 = 0. \tag{1.22}
$$

As $r \to \infty$, since $z \to 0$ exponentially, yx is positive. Therefore, under the assumption that yx does not change sign, the integrands in both integrals in (1.22) are positive definite. We have thus proven the following theorem.

Theorem 1 *Any solution of the field equations* $(1.5 - 1.8)$ *with nonvanishing fields* u, v *must be such that the product* yx *has at least one zero in the open interval* $(0, \infty)$.

We remark here that the theorem is valid also for solutions of the equations linearized with respect to u and v. The derivation in this case parallels that of equations (1.18–1.21) above and leads to an equation like (1.22), except that the v^2 terms are absent from the first integral.

2 Static Solutions for W's in the Background of the $\mathrm{Z_{NO}}$ Vortex.

Recall that, for the $\mathrm{Z_{NO}}$ vortex, the electromagnetic vector potential is zero. In terms of our auxiliary fields x and y, this translates into the condition

$$
\gamma x + (1-\gamma) y = -\frac{2\gamma n + m}{2r} \ .
$$

One then obtains

$$
x = (\gamma - 1) z - \tfrac{2\gamma n + m}{2r} \ , \tag{2.1}
$$
$$
y = \gamma z - \tfrac{2\gamma n + m}{2r} \ , \tag{2.2}
$$

where $z - \frac{n}{r} \equiv Z$ is the vector potential of the Nielsen-Olesen vortex solution.

We shall first investigate the possibility of a perturbative solution about the $\mathrm{Z_{NO}}$ vortex. This means that one should look for solutions of the set of equations (1.2–1.8) to lowest order in the fields u, v. Since the source for A_φ is proportional

to $j(r)$ given by equation (1.14), perturbations in the electromagnetic field do not contribute to this order. Perturbations in s and z will also be quadratic in u, v. The equations for s, x, y will then be the same as for the $\mathrm{Z_{NO}}$ vortex. The equations for u, v are (1.5) and (1.6), linearized with respect to u and v. The boundary conditions for solutions of the linearized equations are the same as the boundary conditions (1.9) and (1.13) of the exact equations.

Let us denote by $\mathcal{Z}_{\mathrm{NO}}$ the point in function space corresponding to the Nielsen-Olesen field configuration of the $\mathrm{Z_{NO}}$ vortex. We shall establish a condition on the possible solutions in the $\mathrm{Z_{NO}}$ background. For this purpose we need the following lemmas.

Lemma 1 *The function $z = y - x$ of $\mathcal{Z}_{\mathrm{NO}}$ is positive definite for all values of the parameter β.*

In fact near $r = 0$, $z \approx \frac{n}{r}$ is positive. As $r \to \infty$, z goes exponentially to zero. Then if z were to change sign it would have to go through a negative minimum. The equation for z in the case of the $\mathrm{Z_{NO}}$ vortex is

$$z'' + \frac{z'}{r} - \frac{z}{r^2} - zs^2 = 0 \ . \tag{2.3}$$

At a negative minimum of z this gives

$$z'' = (\frac{1}{r^2} + s^2)z < 0 \ . \tag{2.4}$$

But this is the condition for a maximum. Therefore z cannot have a negative minimum, hence it cannot change sign.

Lemma 2 *For $\mathcal{Z}_{\mathrm{NO}}$ the value of z_0 in the expansion $z = \frac{n}{r} + z_0 r + \ldots$ near the origin is negative.*

In fact the equation for z can be written

$$\frac{d}{dr}\Big(\frac{1}{r}\frac{d}{dr}(rz)\Big) - zs^2 = 0 \ . \tag{2.5}$$

Integrating from 0 to ∞ one obtains

$$z_0 = -\frac{1}{2}\int_0^\infty zs^2 dr \ . \tag{2.6}$$

Since z is positive definite it follows that $z_0 < 0$. For $\beta = 1,\ z_0 = -1/4$.

We are now ready to prove the following theorem.

Theorem 2 *Any solution of the field equations* $(1.5 - 1.6)$, *with nonvanishing* u, v *and the fields* (s, x, y) *fixed at their* $\mathrm{Z_{NO}}$ *vortex values,* or *these same equations linearized with respect to* u *and* v, *must satisfy the condition* $-2n < m < 0$.

As was already pointed out, Theorem 1 holds true for solutions of the linearized equations as well as of the exact equations. Consider now the asymptotic expressions for y and x,

$$\begin{aligned}
\text{As } r \to 0: \quad & x = -\tfrac{2n+m}{2r} + (\gamma - 1) z_0 r + \mathcal{O}(r^3) \ , \\
& y = -\tfrac{m}{2r} + \gamma z_0 r + \mathcal{O}(r^3) \ , \\
\text{As } r \to \infty: \quad & x = y = -\tfrac{2\gamma n + m}{2r} \ .
\end{aligned}$$

Recall that $z_0 < 0$ for all values of (β, γ).

Assume that the bound on m is not satisfied. Then we have the following two cases:

1) $m \geq 0$. Then x and y are both negative as $r \to 0$ and at large r.
2) $m \leq -2n$. Then x and y are both positive as $r \to 0$ and at large r.

A possible exception occurs for $m = -2n$ and $\gamma = 1$. In this case $x(r) \equiv 0$ and the condition (1.22) for the existence of a solution would require $u(r) \equiv 0$. But then the energy would always increase for any non-vanishing configuration of $v(r)$. For this reason this case is excluded from consideration. In all other cases, if x or y were to change sign, they would have to go through a positive maximum and a negative minimum. But x and y satisfy respectively the equations

$$y'' + \frac{y'}{r} - \frac{y}{r^2} - \gamma z s^2 = 0 \ , \tag{2.7}$$

$$x'' + \frac{x'}{r} - \frac{x}{r^2} - (\gamma - 1) z s^2 = 0 \ , \tag{2.8}$$

where, as already shown, z is positive definite. Then, at a positive maximum of y, one would have

$$y'' = \frac{y}{r^2} + \gamma z s^2 > 0 \ , \tag{2.9}$$

which is the condition for a minimum. At a negative minimum of x

$$x'' = \frac{x}{r^2} + (\gamma - 1) z s^2 < 0 \ , \tag{2.10}$$

which is the condition for a maximum. Therefore neither y nor x changes sign and, by Theorem 1, no solution exists, which proves the theorem.

Let us now apply Theorem 2 to perturbative solutions about the $\mathrm{Z_{NO}}$ vortex. The linearized equations lead to the same constraints (1.10) on m, n, and k as the exact equations for all fields. According to these constraints, a solution of the equations near $r = 0$ is possible only for $m \geq 0$ or $m \leq -2n - 1$. But Theorem 2 tells us that no global solution exists for these values of m. We thus arrive at the following result:

Theorem 3 *In the one-component Higgs gauge, a perturbative solution of the Euler-Lagrange equations about the $\mathcal{Z}_{\rm NO}$ configuration does not exist for any values of β and γ, ($\beta > 0$, $0 \leq \gamma \leq 1$).*

Our field ansatz is ill-defined for $\gamma = 0$, but in this case the physical Z field is aligned with the U(1) hypercharge field and does not couple at all to the W bosons.

An analysis of the variation of the energy at $\mathcal{Z}_{\rm NO}$, done in Ref. [9] for $n = 1$, shows that the energy decreases, in a region of the parameter space (β, γ), for a perturbation in the W field with values $m = -1$ and $k = 0$. This value of k implies that the W production is concentrated at the core of the vortex; this is natural since there the Z field strength takes its maximal value, the Higgs field is minimal, and the vacuum instability due to the anomalous magnetic moment of the W boson [6, 7, 8] is most pronounced. As we have seen, for $n = 1$ the values of m for which the boundary conditions at $r = 0$ for the equations (1.5–1.8) linearized with respect to u, v admit a solution, exclude the values $m = -1, -2$. Nevertheless, within the region (β, γ) of instability of the $\mathrm{Z_{NO}}$ vortex, one expects the energy to have a minimum for some W configuration with $m = -1$ and the fields s, z fixed at their $\mathcal{Z}_{\rm NO}$ values. This minimum would be a solution of the exact equations (1.5) and (1.6) with boundary conditions (1.11.e) and (1.13).

Before proceeding with the investigation of these equations, we shall establish the following result.

Theorem 4 *If, for some values of β, γ, the equations (1.5–1.7) with s, z given by their $\mathcal{Z}_{\rm NO}$ values, admit a solution such that the energy of the corresponding state, calculated in the quadratic approximation in the W field, is lower than that for the $\mathrm{Z_{NO}}$ vortex, then for these values of β and γ the $\mathrm{Z_{NO}}$ vortex is a saddle point in the space of field configurations.*

We shall prove the theorem for the case in which A_φ is allowed to vary and y satisfies equation (1.7) with $\alpha = \gamma$. A similar proof can be given for the other case ($y \equiv y_{\rm NO}$ and $\alpha = 1$).

Suppose that for some values of β, γ, we have a solution of the equations (1.5–1.7) corresponding to the production of W's and an electromagnetic potential A_φ in the $\mathrm{Z_{NO}}$ background. Let u, v, y be the functions corresponding to this solution and write $y = y_{\rm NO} + \overline{y}$, $x = x_{\rm NO} + \overline{y}$, where $x_{\rm NO}, y_{\rm NO}$ are the values of x, y in the $\mathcal{Z}_{\rm NO}$ configuration, given by (2.1, 2.2). Let $\mathcal{E}_0$ be the energy for the $\mathrm{Z_{NO}}$ vortex (with $\Phi_0^2 = 1$ for simplicity) and let us break up the energy difference $\delta\mathcal{E} = \mathcal{E} - \mathcal{E}_0$ into three terms:

$$\mathcal{E}_1 = \frac{2\pi}{\gamma}\int r dr\, [(v' + \frac{v}{r} + 2y_{\rm NO}u)^2 - 4(y'_{\rm NO} + \frac{y_{\rm NO}}{r})uv + \gamma(u^2 + v^2)s^2] \ , \tag{2.11}$$

$$\mathcal{E}_2 = \frac{2\pi}{\gamma}\int r dr\, [4(v'\overline{y} - \overline{y}'v)u + 4(2y_{\rm NO}\overline{y} + v^2)u^2 + \frac{1}{1-\gamma}(\overline{y}' + \frac{\overline{y}}{r})^2] \ , \tag{2.12}$$

$$\mathcal{E}_3 = \frac{2\pi}{\gamma}\int r dr\,(2\overline{y}u)^2\ . \tag{2.13}$$

Since $\overline{y}$ is already second order in (u, v), then $\mathcal{E}_1$ is the lowest order energy shift which is assumed to be negative, $\mathcal{E}_1 < 0$.

Consider configurations of the fields $\overline{y}_\lambda = \lambda\overline{y}, u_\lambda = \sqrt{\lambda}u, v_\lambda = \sqrt{\lambda}v$, where λ is a scaling factor. The energy corresponding to these configurations with s, z at their $\mathcal{Z}_{\rm NO}$ values will be given by

$$\mathcal{E}(\lambda) = \mathcal{E}_0 + \lambda\mathcal{E}_1 + \lambda^2\mathcal{E}_2 + \lambda^3\mathcal{E}_3\ . \tag{2.14}$$

This is a cubic polynomial in λ with extrema at

$$\lambda_\pm = \frac{1}{3\mathcal{E}_3}\Big(- \mathcal{E}_2 \pm \sqrt{\mathcal{E}_2^2 - 3\mathcal{E}_3\mathcal{E}_1}\Big)\ . \tag{2.15}$$

Since $\mathcal{E}_1 < 0$, and $\mathcal{E}_3 > 0$, only λ_+ is a positive root corresponding to a minimum of $\mathcal{E}(\lambda)$. Thus in the interval $0 < \lambda < \lambda_+$ the energy $\mathcal{E}(\lambda)$ will be smaller than $\mathcal{E}(0) = \mathcal{E}_0$. Therefore there will be configurations of arbitrarily small fields $\overline{y}_\lambda, u_\lambda, v_\lambda$ for which the energy becomes smaller than $\mathcal{E}_0$. This proves the theorem.

Now, by construction, $\mathcal{E}(\lambda)$ has an extremum at $\lambda = 1$. Therefore the value of λ at the minimum of $\mathcal{E}(\lambda)$ is $\lambda_+ = 1$. Inserting this value in the expression (2.15), one obtains

$$\mathcal{E}_1 = -(2\mathcal{E}_2 + 3\mathcal{E}_3)\ . \tag{2.16}$$

This is a very useful result for numerical computations since it provides a good check on the precision of the calculation of the energy shift $\delta\mathcal{E}$,

$$\delta\mathcal{E} \equiv \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \frac{1}{2}(\mathcal{E}_1 - \mathcal{E}_3) = -(\mathcal{E}_2 + 2\mathcal{E}_3)\ . \tag{2.17}$$

3 Numerical Solutions

In this section we present solutions of the set of equations (1.5–1.6) for W's in the presence of the fields s, x, y given by their $\mathcal{Z}_{\rm NO}$ values. The second-order equation (1.5) can be written in the form of two first-order equations for the functions rv and $F \equiv v' + \frac{v}{r} + 2uy$, as

$$(rv)' = r(F - 2uy)\ , \tag{3.1}$$

$$F' = (\gamma s^2 + 4u^2)v - 2(y' + \frac{y}{r})u\ . \tag{3.2}$$

Here $y = \gamma Z - \frac{m}{2r}$, where Z and s are given by the Nielsen-Olesen vortex solution and $m = -1$. Equation (1.6) was used to solve algebraically for u in terms of v and F,

$$u = 2\frac{(y' + \frac{y}{r})v - yF}{\gamma s^2 + 4v^2}\ . \tag{3.3}$$

β	γ	v_0	v_1	$\delta\mathcal{E}$
.5	.25	$-.092750$	.06860281	$-.00028$
.5	.5	$-.211671$	.41657161	$-.0106$
1	.5	$-.237408$	.35632791	$-.0133$
.5	1	$-.377157$	1.04195850	$-.0846$
1	1	$-.418631$	.79309235	$-.1003$

Table 1: Boundary values and change of energy due to W-boson condensation in the fixed background of the Z_{NO} vortex, for a selection of parameters β and γ.

The system of equations was first propagated from $r_1 = 18.5$ down to $r_0 = 0$. The boundary condition $v_0 = -u_0$ imposed at this point allowed for a precise determination of v_1, while v_0 was determined with less precision. Values of these two parameters, which determine the boundary conditions at $r = 0$ and at $r = \infty$, are listed in Table 1 for several different values of β and γ. Next, using these boundary values, the equations were propagated from r_1 and from $r_0 = 0$ to a matching point in the interval $(1.5, 2.5)$. The values of u and v agreed within a precision of 10^{-2} at matching points in this interval. Both u and v vary monotonically without changing sign.

Finally, since this is a non-perturbative solution, one has to check whether the energy of these states with W's is actually lower than that of the Z_{NO} vortex. We computed the energy difference between each of these states and the Z_{NO} vortex and found in each case $\delta\mathcal{E} < 0$. Hence in the sector ($\beta > .5$, $\gamma > .25$) we find that the Z_{NO} vortex is unstable against W production. The results for $\delta\mathcal{E}$ in units of Φ_0^2 (energy per unit length of vortex) are also given in Table 1. The precision given for the parameters v_0, v_1, is that required to obtain the precision in the energy shift $\delta\mathcal{E}$. However the accuracy of these numbers is sensitive to the accuracy in the determination of the input functions s, z as solutions of the Z_{NO} vortex.

In each of the cases reported here the relations (2.17) (with $\overline{y} = 0$, $\mathcal{E}_2 = 0$) were satisfied within the accuracy obtained for $\delta\mathcal{E}$.

The line in parameter space along which the equations considered in this section cease to have a solution with $\mathcal{E}_0 < 0$, provides an upper bound on the boundary line Γ for the stability region of the Z_{NO} vortex. An upper bound for this curve has also been obtained by Klinkhamer and Olesen [15] using a different approach.

4 Field Configuration of Minimal Energy with W Fields

Since the $\mathrm{Z_{NO}}$ vortex with $n = 1$ is unstable in a region of the parameter space (β, γ) with respect to W production in a state of angular momentum $m = -1$, and because the energy is bounded from below, one expects that there exists some configuration of the fields, with W's in such a state, for which the energy is a minimum. The Euler-Lagrange equations (1.2–1.8), however, do not admit a solution for $m = -1$ with the boundary conditions required to make the energy finite. This seems to preclude the existence of a stationary state with W's at least defined in the space of differentiable functions. A minimum energy state may nevertheless exist in the sense of a distribution.

In order to investigate this possibility we propose to study a model obtained from Weinberg-Salam (WS) model with the following modifications:

i) Set $\mathbf{W} = \sqrt{\epsilon}\,\mathbf{W}'$

ii) Add to the energy density a term

$$\frac{1}{\gamma}\epsilon\,(1-\epsilon)(i\mathbf{W}'^{\dagger} \times \mathbf{W}')^2$$

This term is positive definite for $0 < \epsilon < 1$. In the new model the boundary conditions at $r = 0$ can be satisfied for $m = -1$.

The model coincides with WS for $\epsilon = 1$. In the instability region of parameter space, the Euler-Lagrange equations for the fields have a solution which, as $\epsilon \to 0$, approaches that found in the WS model for W's in the $\mathrm{Z_{NO}}$ vortex background. As ϵ increases from 0 to 1, either of the following possibilities may occur:

1) The equations do not have a solution for $\epsilon > \epsilon_{\max} < 1$.
2) As $\epsilon \to 1$ the solutions approach almost everywhere the configuration in the vacuum state and the energy approaches zero.
3) As $\epsilon \to 1$ one obtains a sequence of solutions which has no limit but the energy has a definite limit greater than zero.

In cases 1) and 2) a stable vortex with W's and a finite energy does not exist. In case 3) the existence of a stable vortex depends on establishing the stability of the solutions as $\epsilon \to 1$.

Feza Gürsey in Memoriam

At this conference one of us (S.W.M.) presented a special contribution in memory of Feza Gürsey, which will be published elsewhere in these proceedings. The other (O.T.) wishes to express in this space his feelings of gratitude to Feza Gürsey for his inspired teaching of quantum field theory at Yale University and for the many ways graduate students have benefitted, through the years, from his vibrant personality.

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